MHD from a Microscopic Concept and Onset of Turbulence in Hartmann Flow

L. Jirkovsky,1,∗ L. Ma. Bo-ot,2,3,4 and C.M. Chiang3

1Department of Informatics and Geo-informatics, Fakulta Zivotniho Prostredi, University of J.E. Purkyne, Kralova Vysina 7, 40096 Usti n.L., Czech Republic

2National Institute of Physics, University of the Philippines, Diliman, Quezon City, Philippines 1101

3Department of Architecture, National Cheng Kung University, Tainan 701, Taiwan

4College of Architecture, University of the Philippines, Diliman, Quezon City, Philippines 1101

(Received March 31, 2009)

Abstract We derive higher order magneto-hydrodynamic (MHD) equations from a microscopic picture using projection and perturbation formalism. In an application to Hartmann flow we find velocity profiles flattening towards the center at the onset of turbulence in hydrodynamic limit. Comparison with the system under the effect of a uniform magnetic field yields difference in the onset of turbulence consistent with observations, showing that the presence of magnetic field inhibits onset of instability or turbulence. The laminar-turbulent transition is demonstrated in a phase transition plot of the development in time of the relative average velocities vs. Reynolds number showing a sharp increase of the relative average velocity at the transition point as determined by the critical Reynolds number.

PACS numbers: 52.30.Cv, 05.70.Fh, 52.35.Ra, 52.65.Kj
Key words: projection techniques, onset of turbulence, MHD, Hartmann flow

1 Introduction

Earlier, we developed a modified momentum transport equation for neutral fluids using a perturbation approach and applied the equation to describe the laminar-turbulent transition for the circular and flat plate systems.[1−3] In this paper we study extending of the equation and its use to electro-magnetically conducting fluids, otherwise known as magneto-hydrodynamics or MHD.

The standard MHD equations can be solved analytically if applied to some of MHD flows with the theory in good agreement with experiments as long as the flow remains laminar. However, the standard MHD equations encounter difficulty whenever there is a onset of turbulence.[4] Justification of local classical transport in the face of observed dissipative processes, like turbulence, has not reached full closure.[5] The reason may be found in the traditional assumption about hard-sphere structure-less particles used in the derivation. Elastic collisions between the particles of the fluid do not affect the form of standard MHD equations, which can be derived from Boltzmann kinetic equation as the contribution from Boltzmann collision integral reduces to zero. It might be therefore useful to study the onset and origin of turbulence directly from the more general Liouville equation. Adopting and reformulating the projection and perturbation analyses for MHD from Refs. [1−3] extended to the third and higher order, kinetic equations for single particle distribution functions and relevant MHD equations containing two control parameters related to the internal structure and geometry of the particles are derived. We employ the macroscopic classical approach using the microscopic quality of quantum mechanics resulting in a modification of the MHD equations.

If one adopts a hypothesis about a microscopic origin of turbulence, there are the dissipative effects because of excitations of internal degrees of freedom of the particles due to collisions resulting in inelastic interactions.[6] This is reflected in correction terms appearing in higher order MHD equations. Examples on excitations of internal degrees of freedom of the particles are transitions between rotational levels inducing the occurrence of inelastic interactions and irregular motion. Monatomic gas is modeled as a quantum confinement with discrete energy levels. Electrons, as particles with no internal structure, may undergo inelastic interactions as well because of existence of two spin states.

Thus, on a microscopic scale the inelastic interactions are simply result of events on the quantum level. Such a quantum kinetic model of turbulence is based on the well-known idea associating turbulence with deterministic chaos induced by inelastic interactions. Experimental evidence have been reported suggesting the existence of the slight difference in critical Reynolds numbers for different gases as predicted by the theory, particularly for carbon monoxide and nitrogen[6−8] due to the different energy gaps between ground and first excited rotational levels.

For Hartmann flow, the only non-vanishing correction terms are those with control parameter related to the internal quantum structure of the particle. The time development of the velocity profiles is obtained through numeric simulation of the macroscopic equations obtained purely in classical way. For sufficiently high value of the control parameter the parabolic velocity profile exhibits flattening towards the center, manifesting the onset of instability or turbulence in hydrodynamic limit.[1−2,8] On
the other hand in uniform magnetic field, although the velocity profile is flattened due to the \( c/m \) brake in laminar regime, no change in profiles is observed for the comparable value of the control parameter, in accordance with observations that magnetic field inhibits turbulence.\(^4\) A much higher control parameter is needed to induce turbulence.

2 Preliminaries

We consider a plasma fluid consisting of an equal number of negatively charged electrons and positively charged ions such that the total number of particles in the system is \( N \). The Liouville equation for the \( N \)-particle plasma distribution function is

\[
\frac{\partial f^{(N)}}{\partial t} = -L f^{(N)}.
\]

The Liouville operator can be written as \( L = L_0 + L_{ec} + L_i \) with coupling strength parameter \( \lambda \in (0, 1) \) and the operators of the kinetic, external potential, and interaction energies as

\[
L_0 = -i \sum_j \vec{p}_j \frac{\partial}{\partial \vec{x}_j}, \quad L_{ec} = -i \sum_j \vec{F}_j(\vec{p}_j) \frac{\partial}{\partial \vec{p}_j}, \quad L_i = \frac{1}{2} i \sum_{j \neq j'} \vec{\nabla}_j V(\vec{x}_j - \vec{x}_j') (\vec{\nabla}_{p_j} - \vec{\nabla}_{p_{j'}}),
\]

where \( \vec{p}_j, m_j \) are the momentum and mass of the \( j \)-th particle, \( \vec{F}_j \) are slowly varying velocity dependent forces and \( V \) represents interaction potential energy.

Following [1–3, 9], define projectors

\[
P = \frac{1}{\Omega^{N-1}} \int d\vec{x}_2 \cdots d\vec{x}_N
\]

with \( \Omega \) as the volume of the system and \( Q = 1 - P \). Then \( f_P = Pf^{(N)} \) and \( f_Q = Qf^{(N)} \) are reduced plasma distribution functions. Application of the projectors to both sides of the Liouville equation yields the system of two equations

\[
\begin{align*}
\frac{\partial f_P}{\partial t} &= PL f_P + PL f_Q, \quad (2) \\
\frac{\partial f_Q}{\partial t} &= QL f_P + QL f_Q. \quad (3)
\end{align*}
\]

The formal solution of Eq. (3) is

\[
f_Q(t) = G(t) f_Q(0) - i \int_0^t G(t-s)QL f_P(s) ds. \quad (4)
\]

We call \( G(t) = \exp(-i \int f QLds) \) a propagator. The substitution of Eq. (4) into Eq. (2) and integrating over entire momentum space but one particle gives us an exact kinetic equation for single particle distribution function \( f \)

\[
\frac{\partial f}{\partial t} = -i(\lambda - 1) f - \lambda L f + \lambda L f \quad (5)
\]

Following a perturbation method developed in [2–3], expand the single particle distribution function \( f \) in orders of coupling parameter \( \lambda \)

\[
f = \sum_{k=0}^{\infty} \lambda^k f^{(k)} \quad (6)
\]

and the propagator \( G \) in Taylor series

\[
G = \exp(-iQLt) = \sum_{n=0}^{\infty} \frac{1}{n!}(-iQLt)^n. \quad (7)
\]

Then we substitute the appropriate expansions of \( f \) and \( G \) into Eq. (5) and pick the terms containing same powers of \( \lambda \) to get kinetic equations of appropriate orders. This approximation is possible for sufficiently small time \( t \).

Zero order, \( (k = 0) \) We obtain Boltzmann equation for single particle distribution function \( f^{(0)} \) with BBGKY-like elastic collision term decoupled from time derivative of the zero order distribution function

\[
\frac{\partial f^{(0)}}{\partial t} + \vec{p} \vec{\nabla}_f f^{(0)} + F \vec{\nabla}_p f^{(0)} = \left( \frac{\partial f}{\partial t} \right)_{\text{coll}}. \quad (8)
\]

The Lorentz force acting on a charged particle is \( \vec{F} = \psi \vec{E} + \vec{p} \times \vec{B}/m \). Multiplying Eq. (8) by one component of momentum and integrating over momentum space we get standard MHD equation for the mean velocity of the fluid

\[
\rho^{(0)} \left( \frac{\partial U_i^{(0)}}{\partial t} + U_j^{(0)} \frac{\partial U_i^{(0)}}{\partial x_j} \right) + \frac{\partial P_{ij}^{(0)}}{\partial x_j} - (\vec{j} \times \vec{B})_i = 0. \quad (9)
\]

For an incompressible fluid we get

\[
\rho^{(0)} \left( \frac{\partial U_i^{(0)}}{\partial t} + (\vec{U}_j^{(0)} \vec{\nabla}) \vec{U}_i^{(0)} \right) + \vec{\nabla} P^{(0)} - \rho^{(0)} \nu \vec{\nabla}^2 \vec{U}_i^{(0)} - \vec{j} \times \vec{B} = 0. \quad (10)
\]

We use a decomposition of the pressure tensor \( P_{ij} \) into its diagonal \( (\rho \delta_{ij}) \) and off-diagonal parts \( (\sigma_{ij}, i, j = 1, 2, 3 \), the shear stress). The zero order density, pressure and kinematic viscosity are classic definitions from kinetic theory and the contribution from the elastic collision term reduces to zero. Here we consider fully ionized plasma with isotropic viscosity and neglect the effects of anisotropy due to the presence of a magnetic field.\(^{[10]}\)

First order, \( (k = 1) \) We obtain MHD equation with a correction term \( \partial_j \left( \rho U_i U_j \right)_t \), known as the Reynolds MHD equation using the procedure in \([9]\)

\[
\rho^{(0)} \left( \frac{\partial U_i^{(0)}}{\partial t} + (U_j^{(0)} \partial_j) U_i^{(0)} \right) + \partial_j P^{(0)} - \rho^{(0)} \nu \partial_j^2 U_i^{(0)} - (\vec{j} \times \vec{B})_i + \partial_j (U_i^{(1)} U_j^{(1)})_t = 0. \quad (11)
\]

The first order velocity components are interpreted as fluctuations. Fourier transformed form is used in most theories of turbulence. Although turbulence is traditionally studied from the Reynolds equation, it is treated as a flow phenomenon and the particle structure is unimportant or unnecessary. Recognizing the limitations of the Reynolds equation in explaining the onset and origin of turbulence, both of which are still open problems, we explore the possibilities of higher order kinetic and MHD equations.
Second order, \((k = 2)\) We pick terms containing \(\lambda^2\) to get the second order kinetic equation

\[
\frac{\partial f^{(2)}}{\partial t} + i(L_0 + L_c)f^{(2)} = -PL \int_0^t [1 - i(t-s)(L_0 + L_c) - \frac{1}{2}(t-s)^2(L_0 + L_c)^2]L_if^{(2)}(s)ds .
\] (12)

Using the definitions of \(P, L_0, L_i,\) and \(L_c\) we rewrite Eq. (12) as

\[
\frac{\partial f^{(2)}}{\partial t} + \frac{\vec{p}}{m} \vec{\nabla} f^{(2)} + F \vec{\nabla}_p f^{(2)} = \int_0^t b(\vec{\nabla}_p f^{(0)})(s)ds - \frac{1}{m} \vec{\nabla} \int_0^t b(t-s)^2(\vec{\nabla}_p + \vec{\nabla}_p^2)f^{(0)}(s)ds + \frac{1}{2m} \vec{\nabla}^2 \int_0^t b(t-s)^2(2\vec{p} \vec{\nabla}_p + \vec{\nabla}_p^2 f^{(0)}(s)ds ,
\] (13)

where parameter \(b = \int_\Omega (\vec{\nabla} V)^2 d\Omega\) is interpreted as a strength of inelastic interactions. It vanishes for elastic collisions. After we multiply Eq. (13) by one component of the momentum and integrating over momentum space, the third order MHD equation is

\[
\text{Third order, } (k = 3) \text{ We pick terms containing } \lambda^3. \text{ With explicit forms of the Liouville operators the third order kinetic equation reduces to}
\]

\[
\frac{\partial f^{(3)}}{\partial t} + \frac{\vec{p}}{m} \vec{\nabla} f^{(3)} + \frac{\vec{F}}{m} \vec{\nabla}_p f^{(3)} = -\frac{1}{m} \int_0^t b(t-s)[\vec{\nabla}_p \vec{\nabla} + \vec{\nabla}_p^2 \vec{\nabla}]f^{(1)}(s)ds + \frac{2}{3m^2} \vec{\nabla}^3 \int_0^t b(t-s)^3[3\vec{p} \vec{\nabla}_p + \vec{\nabla}_p^2 f^{(1)}(s)ds - \frac{2}{3m^2} \vec{\nabla}^3 \int_0^t b(t-s)^2 \vec{\nabla}_p f^{(1)}(s)ds .
\] (17)

The vector parameter \(\vec{c} = \int_\Omega (\vec{\nabla} V)^3 d\Omega\) is related to the geometry of the particle and is interpreted as a measure of the asymmetry of the particle. It vanishes for perfectly spherical particle. Multiplying Eq. (17) in component form by a component of the momentum and integrating over momentum space, the third order MHD equation is

\[
\rho \left[ \frac{\partial \vec{U}}{\partial t} + (\vec{U} \vec{\nabla}) \vec{U} \right] + \vec{\nabla} \rho - \rho \vec{V} \vec{\nabla} \vec{U} + \vec{\nabla} \times \vec{B} = -\frac{1}{m} \int_0^t b(t-s) \vec{\nabla} \rho - (t-s)^2 \vec{\nabla}^2 \vec{U} ds
\]
We re-normalize the density, pressure, and velocity to their true values.

**Higher order, \( k > 3 \)** Second and third order MHD equations may be written in a compact form by introducing a new dimensionless parameter \( \varepsilon \in (0, 1) \)

\[
\rho \left[ \frac{\partial \tilde{U}}{\partial t} + \tilde{(U \cdot \nabla)} \tilde{U} \right] + \tilde{\nabla} p - \nu \rho \tilde{\nabla}^2 \tilde{U} - \tilde{j} \times \tilde{B} = - \frac{1}{m} \int_0^t b(t - s) \tilde{\nabla} \rho - (t - s)^2 \tilde{\nabla}^2 \tilde{U} \bigg|_{ds}
\]

\[- \varepsilon \frac{4}{m^2} \int_0^t (t - s)^3 [5\tilde{c}\tilde{\nabla}^2 \rho + b\tilde{\nabla}^3 \tilde{U}] ds. \] (19)

When \( \varepsilon = 0 \) we have the second-order equation, while for \( \varepsilon = 1 \) we have the third-order equation. For a value of \( \varepsilon \) between 0 and 1, we think the Eq. (19) is a good approximate of higher order equations. A justification comes besides from its utility from the fact the Taylor expansion of the propagator \( G \) is an alternating series. As a result the time developments of the mean velocity profiles of the flow are oscillating around the solution of the exact transport equation as one raises the order of the approximate transport equation. The oscillations diminish with raising order. Specifically for the Hartmann flow in hydrodynamic limit, mean velocities are decreasing in even orders while increasing in odd orders in time as seen in numeric simulations of the second and third order equations.[2] We note that the parameter \( \tilde{c} \) does not affect the flow of an incompressible fluid since the density \( \rho \) is uniform. This reduces Eq. (19) to

\[
\rho \left[ \frac{\partial \tilde{U}}{\partial t} + \tilde{(U \cdot \nabla)} \tilde{U} \right] + \tilde{\nabla} p - \nu \rho \tilde{\nabla}^2 \tilde{U} - \tilde{j} \times \tilde{B} = \frac{b}{m} \int_0^t (t - s)^2 \tilde{\nabla}^2 \tilde{U} ds - \frac{4\varepsilon b}{m^2} \int_0^t \tilde{\nabla}^3 \tilde{U}^2 ds. \] (20)

### 3 Application to Hartmann Flow

After following in principle the procedure described in [4], Eq. (20) when coupled with Maxwell equations and differentiated four times with respect to time \( t \) to eliminate the time integrals can be reduced to a fifth order nonlinear differential equation for Hartmann flow. With the short-circuit condition and configuration shown in Fig. 1 we have

\[
\frac{\rho \partial^5 U}{\nu \partial t^5} + \frac{\sigma B_0^2 \partial^3 U}{\nu c^2 \partial t^3} - \frac{\rho \partial^6 U}{m \nu \partial t^3 \partial x^2} - \frac{2b}{m \nu} \frac{\partial^3 U}{\partial x^2} \frac{\partial^3 U}{\partial x^2} + \frac{4\varepsilon b}{m^2 \nu} \left( \frac{\partial^2 U}{\partial x^2} \frac{\partial^2 U}{\partial x^2} + \frac{U \partial^3 U}{\partial x^3} \right) = 0. \] (21)

The parameter \( \varepsilon \) can be fine adjusted in numeric simulations.

The set-up for the Hartmann flow is the simplest geometry studied in the development of MHD generators and plasma propulsion. It also illustrates the basic mechanism of a number of applications of MHD flows. The system consists of a conducting plasma body made to move along a rectangular duct of constant cross section.

A pair of opposite enclosing walls at distance \( L \) is electrically conducting while the others are insulators. A schematic illustration is shown above – Fig. 1.

The induced electric current interacts with the external magnetic field and the resulting Lorentz force acts always against the flow. This is the case of MHD generators. When the current is maximum the \( e/m \) brake is also maximum but the electric field is zero because of short-circuit condition. Given a finite external resistance, electric power can be extracted from the plasma.

![Fig. 2 Velocity profiles for Hartmann flow. U vs. x, b = 0.001, ε = 0.000 005, B = 5, t = 0 – 10, bottom to top, near coincidence of velocity profiles for t = 2 – 10.](image2)

![Fig. 3 Velocity profiles for Hartmann flow. U vs. x, b = 0.001, ε = 0.000 005, B = 0, t = 0 – 10, bottom to top.](image3)
c = 1, $\sigma = 1$. The Reynolds number is $R_n = U_{max} l / \nu$ with characteristic length $l = 0.5 L$. For the flow along a rectangular duct in the hydrodynamic limit, the results shown in Fig. 3 exhibit the onset of instability and flattening of velocity profiles starting with control parameter $b = 0.001$. On the other hand velocity profiles for $e/m$ brake with the same control parameter remain unchanged as in Fig. 2, suggesting that a higher control parameter is needed to induce laminar-turbulent transition (See Figs. 4 and 5). This is consistent with the observation that magnetic field inhibits or delays the onset of turbulence.$]^{[4]}$

![Fig. 4](image)

**Fig. 4** Velocity profiles for Hartmann flow. $U$ vs. $x$, $b = 0.001, \varepsilon = 0.000005, B = 0, t = 0-10$, bottom to top, $t = 5-10$, top to bottom, near coincidence of velocity profiles at the center for $t = 2-10$.

The laminar-turbulent transition is demonstrated in a phase transition plot where the time development of the relative average velocities $U_{ave}/U_{max}$ is plotted vs. $\log R_n$ for $b = 0.01$ and $B = 5$ in Fig. 5. There is a sharp increase of the relative average velocity at the transition point determined by the critical Reynolds number. The behavior shown in Fig. 5 is reminiscent of the classical friction factor vs. the Reynolds number plots however the friction factor is based on mean system pressure$]^{[11-12]}$ and there is no hypothesis behind the mechanism to transition as stated by the authors.$]^{[12]$

![Fig. 5](image)

**Fig. 5** Phase transition in Hartmann flow, $U_{ave}/U_{max}$ vs. $\log R_n$, (6 points on horizontal branch, 4 points on vertical branch), $b = 0.01, \varepsilon = 0.000005, B = 5$.

4 Conclusion

In closing, we remark the control parameter $b$ manifests its presence in $e/m$ brake where the centerline velocity decreases with time, the flattening of the profiles broaden and intersect symmetrically at two points as seen in Fig. 4. More importantly, in Fig. 5 there appears to be a bifurcation pointing to laminar-turbulent transition. This is not surprising since the Eq. (21) is nonlinear fifth order differential equation. Its solution can have at most five incommensurable frequencies raising the prospect of turbulent-like behavior in time.$]^{[5]}$ This was proposed before from studies of kinetic equations taken to higher orders in coupling parameter and this study represents an independent motivation, an elaboration, as well as an extension to MHD flows, of such ideas.

References